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A Metric Allowing a 2-Parameter Group of Motion

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Abstract

Using heuristic arguments, we find a metric form, which might describe the field of 'particles' having light velocity. It turns out that this metric, which allows a 2-parameter group of motion, indeed includes radiative solutions of the Einstein vacuum equations and of the Einstein–Maxwell equations.

Latin suffixes range from 1 to 4 and the signature of the metric form is +2. The units are chosen to make the speed of light and the gravitational constant both unity. Commas are used to denote partial derivatives.

Imagine r, ϕ, z to be quasi-cylindrical coordinates and let t be the time. The field of a particle moving with light velocity along the z-axis is probably not 'aging'. We thus expect (i.e. define) the metric g_{ik} to be invariant under the coordinate transformation

$$z' = z + \lambda, \qquad t' = t + \lambda, \qquad r' = r, \qquad \phi' = \phi \tag{1}$$

where λ is a continuous parameter. We furthermore require for simplicity rotational symmetry, i.e. we demand invariance under the transformations

$$z' = z, \qquad t' = t, \qquad r' = r, \qquad \phi' = -\phi \tag{2a}$$

and

$$z' = z, \qquad t' = t, \qquad r' = r, \qquad \phi' = \phi + \mu \tag{2b}$$

where μ is a continuous parameter. Killing's equations lead to

$$g_{ik} = \begin{pmatrix} \alpha & 0 & 0 & \omega \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ \omega & 0 & 0 & \delta \end{pmatrix} \tag{3}$$

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as the simplest form of the metric, with g_{ik} depending on x_3 and x_4 only, where x_i are the new coordinates.

The Ricci tensor for the metric (3) has been obtained and checked on the computer with 'FORMAC'. To obtain vacuum solutions and electromagnetic solutions of the Einstein equations, we make simplifying assumptions. We find indeed radiative solutions, thus justifying the heuristic argument by which (3) was chosen.

The linearized Einstein equations for vacuum, which have been obtained by expanding each term in the Einstein vacuum equations with respect to deviations from flat space

$$\alpha = \delta = 0, \qquad \omega = \gamma = 1, \qquad \beta = x_3^2 \tag{4}$$

and neglecting terms of second or higher order in these deviations, have the following solution:

$$\begin{aligned} \alpha = c \ln x_3 + B, \qquad \omega = 1 + \tfrac{1}{4} x_3^2 B_{,44}, \qquad \gamma = 1 + \tfrac{1}{2} x_3^2 B_{,44} + U_{,33} \\ \delta = -\tfrac{1}{32} x_3^4 B_{,4444} - U_{,44} + V \ln (x_3 + U_{,3}), \qquad \beta = x_3^2 + x_3 U_{,3} \end{aligned} \tag{5}$$

where $B(x_4)$, $V(x_4)$, $U(x_3,x_4)$ are arbitrary functions of their arguments and c is a constant. If we put $B = U = V = 0$ in (5) we have an exact vacuum solution, which is Tolmann's (Peres, 1960) field of a pencil of light. If we put $c = B = 0$ in (5) we have an approximate solution whose corresponding exact solution is the cylindrically symmetric plane-fronted gravitational wave. No exact solution similar to the approximate solution $c = U = V = 0$ in (5) has been found.

Looking for electromagnetic solutions of the form (3) with $\omega = 0$, $\gamma = -\delta = 1$, $\alpha = \alpha(x_3)$, $\beta = \beta(x_4)$, we find

$$-R_1^{\ 1} = -R_3^{\ 3} = R_2^{\ 2} = R_4^{\ 4} = k^2$$

$$\alpha = a \sin^2 k x_3, \qquad \beta = b \sin^2 k x_4, \qquad a, b, k \ldots \text{const} \tag{6}$$

which differs from an Einstein space (Petrov, 1969, equation (14.8)) only by the location of a minus sign in the metric. The non-vanishing components of the electromagnetic tensor F_{ik} are given by

$$F_{13} = c\alpha^{1/2}, \qquad F_{24} = d\beta^{1/2}; \qquad c, d \ldots \text{const}; \qquad 4\pi(c^2 + d^2) = k^2 \tag{7}$$

This corresponds to a constant electric and magnetic field, as the physical components of F_{ik} (Synge, 1964, p. 355) turn out to be

$$F_{\langle 13 \rangle} \equiv -H_2 = c; \qquad F_{\langle 42 \rangle} \equiv -E_2 = d \tag{8}$$

Looking for vacuum and electromagnetic solutions of the form (3) with

$$\alpha = \delta = 0 \qquad \text{and} \qquad R_2^{\ 2} = R_3^{\ 3} \tag{9}$$

we found the vacuum solution

$$\alpha = \delta = 0, \qquad \omega = x_3^{4/3}, \qquad \gamma = x_4^{6/5}, \qquad \beta = x_3^{-2/3} x_4^{-2/5} \tag{10}$$

and the electromagnetic solution

$$\alpha = \delta = 0, \qquad \omega = u^2, \qquad \gamma = 1, \qquad \beta = (u_{,3})^2 \tag{11}$$

where $u(x_3)$ is the solution of the cubic equation

$$u^3 - 9u^2\frac{a}{b} + 108\left(\frac{a}{b}\right)^3 - bx_3{}^2 = 0; \qquad a, b \ldots \text{const} \tag{12}$$

The non-vanishing components of the electromagnetic tensor F_{ik} are given by

$$F_{14} = d, \qquad F_{23} = cu^{-2}u_{,3}; \qquad c, d \ldots \text{const}; \qquad 3\pi(d^2 + c^2) = -a \tag{13}$$

Using an orthonormal tetrad, we find the physical components of F_{ik} to be

$$F_{\langle 14\rangle} \equiv E_1 = -du^{-2}, \qquad F_{\langle 23\rangle} \equiv H_1 = cu^{-2} \tag{14}$$

If we put $a = 0$ in (12), then we obtain the known (Petrov, 1969, equation (26.32)) vacuum solution

$$\alpha = \delta = 0, \qquad \omega = x_3^{4/3}, \qquad \gamma = 1, \qquad \beta = x_3^{-2/3} \tag{15}$$

References

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